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Magneto-thermoelastic Response in a Functionally Graded Isotropic Unbounded Medium Under a Periodically Varying Heat Source

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Abstract This paper deals with the problem of magneto-thermoelastic interactions in a functionally graded isotropic unbounded medium due to the presence of periodically varying heat sources in the context of linear theory of generalized thermoelasticity with energy dissipation (TEWED) and without energy dissipation (TEWOED) having a finite conductivity. The governing equations of generalized thermoelasticity (GN model) for a functionally graded material (FGM) under the influence of a magnetic field are established. The Laplace–Fourier double transform technique has been used to get the solution. The inversion of the Fourier transform has been done by using residual calculus, where poles of the integrand are obtained numerically in a complex domain by using Leguerre's method and the inversion of the Laplace transformation is done numerically using a method based on a Fourier series expansion technique. Numerical estimates of the displacement, temperature, stress, and strain are obtained for a hypothetical material. The solution to the analogous problem for homogeneous isotropic materials is obtained by taking a suitable non-homogeneous parameter. Finally, the results obtained are presented graphically to show the effect of a non-homogeneous, magnetic field and damping coefficient on displacement, temperature, stress, and strain.

Keywords Functionally graded material · Green–Naghdi model · Magneto-thermoelasticity

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List of Symbols

$\mathbf{u}$	Displacement vector
λ, μ	Lamé constants
ρ	Constant mass density of the medium
γ	Thermal module
α_t	Coefficient of linear thermal expansion
T_0	Uniform reference temperature
T	Small temperature increase above the reference temperature T_0
$\mathbf{J}$	Electric current density vector
$\mathbf{B}$	Magnetic induction vector
c_v	Specific heat of the medium at constant strain
K^*	A material constant characteristic for the G–N theory
$\mathbf{H}$	Total magnetic field vector at any time
$\mathbf{E}$	Electric field vector
μ_e	Magnetic permeability of the medium
σ	Electric conductivity of the medium
c_T	Non-dimensional finite thermal wave speed of G–N theory of thermoelasticity II
ϵ_T	Thermoelastic coupling constant
K	Thermal conductivity
κ	Thermal diffusivity

1 Introduction

The classical theories of thermoelasticity, involving an infinite speed of the propagation of thermal signals, contradict physical facts. During the last three decades, non-classical theories involving a finite speed of heat transport in elastic solids have been developed to remove this paradox. In contrast to the conventional coupled thermoelasticity theory, which involves a parabolic type heat transport equation, these generalized theories involving a hyperbolic-type heat transport equation and are supported by experiments exhibiting the actual occurrence of wave-type heat transport in solids, called the second-sound effect. The extended thermoelasticity theory (ETE) proposed by Lord and Shulman [1] incorporates a flux-rate term into Fourier's law of heat conduction and formulates a generalized form that involves a hyperbolic type heat transport equation with a finite speed of the thermal signal. Green and Lindsay [2] developed a temperature-rate dependent thermoelasticity (TRDTE) theory by introducing relaxation time factors that do not violate the classical Fourier law of heat conduction, and this theory also predicts a finite speed for heat propagation. Because of the experimental evidence in support of the finiteness of the speed of propagation of a heat wave, generalized thermoelasticity theories are more realistic than conventional thermoelasticity theories in dealing with practical problems involving very short time intervals and high heat fluxes like those occurring in laser units, energy channels, nuclear reactors, etc.

The phenomenon of coupling between the thermo-mechanical behavior of materials and the electromagnetic behavior of materials has been studied since the nineteenth century. By the middle of the twentieth century, piezoelectric materials were finding their first applications in hydrophones. In the last two decades, the concept of electromagnetic composite materials has arisen. Such composites can exhibit field coupling that is not present in any of the monolithic constituent materials. These so called "Smart" materials and composites have applications in ultrasonic imaging devices, sensors, actuators, transducers, and many other emerging components. Magneto-electro-elastic materials are used in various applications. Due to the ability of converting energy from one kind to another (among mechanical, electric, and magnetic energies), these materials have been used in high-tech areas such as lasers, supersonic devices, microwave, infrared applications, etc. Furthermore, magneto-electro-elastic materials exhibit coupling behavior among mechanical, electric, and magnetic fields and are inherently anisotropic. Problems related to the wave propagation in thermoelastic or magneto-thermoelastic solids using these generalized theories have been studied by several authors. Among them, Paria [3] has presented some ideas about magneto-thermoelastic plane waves. Neyfeh and Nemat-Nasser [4,5] have studied thermoelastic waves and electro-magneto-elastic waves in solids with a thermal relaxation time. Roychoudhuri and Chatterjee(Roy) [6] have introduced a coupled magneto-thermoelastic problem in a perfectly conducting elastic half-space with thermal relaxation. Hsieh [7] has considered modeling of new electromagnetic materials. Ezzat [8] has studied the state space approaches to generalized magneto-thermoelasticity with two relaxation times in a perfectly conducting medium. Ezzat et al. [9] have studied electro-magneto-thermoelastic plane waves, with thermal relaxation in a medium of perfect conductivity. Problems related to magneto-thermoelasticity with thermal relaxation have been investigated by Sherief and Yoset [10] and by Baksi and Bera [11].

Green and Naghdi [12] developed three models for generalized thermoelasticity of homogeneous isotropic materials, which are labeled as models I, II, and III. The nature of these theories is such that when the respective theories are linearized, Model I reduces to the classical heat conduction theory (based on Fourier's law). The linearized versions of models II and III permit propagation of thermal waves at a finite speed. Model II, in particular, exhibits a feature that is not present in other established thermoelastic models as it does not sustain dissipation of thermal energy (Green and Naghdi [13]). In this model, the constitutive equations are derived by starting with the reduced energy equation and by including the thermal displacement gradient among other constitutive variables. Now the GN II model is employed to study the propagation of magneto-thermoelastic waves which do not undergo both attenuation and dispersion and which has been investigated by Roychoudhuri [14]). Green–Naghdi's third model admits the dissipation of energy. In this model, the constitutive equations are derived by starting with the reduced energy equation, where the thermal displacement gradient in addition to the temperature gradient, are among the constitutive variables. Green and Naghdi [15] included the derivation of a complete set of governing equations of a linearized version of the theory for homogeneous and isotropic materials in terms of the displacement and temperature fields and a proof of the uniqueness of the solution for the corresponding initial boundary value problem. In the context of a linearized version of this theory (Green and Naghdi [13,15]), a theorem on uniqueness of solutions has been established by Chandrasekhariah [16,17]. Chandrashekhariah et al. [18] have studied one-dimensional thermal wave propagation in a half-space based on the GN model due to the sudden exposure of the temperature to the boundary using the Laplace transform method. Chandrasekhariah et al. [19] have studied thermoelastic interactions caused by a continuous heat source in a homogeneous isotropic unbounded thermoelastic body by employing the linear theory of thermoelasticity without energy dissipation (TEWOED).

Mallik and Kanoria [20,21] have studied the thermoelastic interaction in an infinite rotating elastic medium in the presence of heat sources in generalized thermoelasticity. The problems have been solved by applying an Eigenvalue approach. Kar and Kanoria [22,23] have analyzed thermoelastic interactions with energy dissipation in a transversely isotropic thin circular disc and in an unbounded body with a spherical hole. Taheri et al. [24] employed Green–Naghdi theories of type II and type III to study thermal and mechanical waves in an annular domain. Roychoudhuri and Dutta [25] have studied thermoelastic interaction in an isotropic homogeneous thermoelastic solid containing time-dependent distributed heat sources which vary periodically for a finite time interval in the context of TEWOED. Bandyopadhyay and Roychoudhuri [26] have considered one-dimensional wave propagation in a homogeneous isotropic thermoelastic half-space using Green–Naghdi model II under various boundary conditions and obtained a short time solution for displacement, temperature, stress, and strain.

The functionally graded material concept originated in Japan in 1984 during the space-plane project in the form of a proposed thermal barrier material. A functionally graded material (FGM) is a two-component composite characterized by a compositional gradient from one component to the other. In contrast, traditional composites are homogeneous mixtures and they, therefore, involve a compromise between the desirable properties of the composite materials. Since significant properties of an FGM contain the pure form of each component, the need for compromise is eliminated. The properties of both components can be fully utilized. The use of FGMs can eliminate or control thermal stresses in structural components (Wetherhold and Wang [27]).

Shankar and Tzeng [28] have analyzed the two-dimensional thermal stress problem for a functionally graded beam whose thermoelastic constants vary exponentially through the thickness. Vel and Batra [29] and Qian and Batra [30] have analyzed the three-dimensional steady-state or transient thermal stress problems of a functionally graded rectangular plate whose material properties vary with a power product form of the thickness. On the other hand, since shell type structures are used in various industrial fields, the thermoelastic analysis of a circular cylinder, sphere, and cylindrical panels made of an FGM becomes important. Lutz and Zimmerman [31,32] have presented the exact solution for one-dimensional thermal stresses of a functionally graded sphere and cylinder whose elastic modulus and coefficient of linear thermal expansion vary linearly with radius. Ye et al. [33] have presented the exact solution for the axisymmetric thermoelastic problem of a uniformly heated functionally graded transversely isotropic cylindrical shell, assuming that the modulus of elasticity and the coefficient of linear thermal expansion vary with the power product form of the radial coordinate variable. El-Naggar et al. [34] have analyzed transient thermal stresses in a rotating non-homogeneous orthotropic hollow cylinder using a finite difference method. Wang and Mai [35] have analyzed the transient one-dimensional thermal stresses in non-homogeneous materials such as plates, cylinders, and spheres using a finite element method. Ootao and Tanigawa [36] have studied exactly a one-dimensional transient thermoelastic problem of a functionally graded hollow cylinder whose thermal and thermoelastic constants are assumed to vary with the power product form of the radial coordinate variable. Shao et al. [37] have solved a thermomechanical problem of an FGM hollow circular cylinder whose material properties are assumed to be temperature independent and vary continuously in the radial direction. Hosseini Kordkheili and Naghbadi [38] have studied the thermoelastic analysis of a functionally graded cylinder under axial loading. The transient thermoelastic problem of a functionally graded cylindrical panel due to non-uniform heat supply has been solved by Ootao and Tanigawa [39]. Bagri and Eslami [40] have analyzed a unified generalized thermoelasticity formulation and application to thick functionally graded cylinders. Analysis of thermoelastic waves in a functionally graded hollow sphere based on the Green–Lindsay theory has been studied by Bagri and Eslami [41]

Nayfeh and Nemat-Nasser [42] have studied electromagnetic thermoelastic plane waves in solids with thermal relaxation. Rakshit and Mukhopadhyay [43] have introduced an electro-magneto-thermo-visco-elastic problem in an infinite medium with a cylindrical hole. Again Tianhu and Shirong [44] have studied a two-dimensional generalized electro-magneto-thermoelastic problem for a half space. Baksi et al. [45] have studied the magneto-thermoelastic problems with thermal relaxation and heat sources in a three-dimensional infinite rotating elastic medium. Roychoudhuri and Chattopadhyay [46] have explained electro-magneto-thermo-visco-elastic plane waves in rotating media with thermal relaxation. Furthermore, thermoelastic interactions with energy dissipation in an infinite solid with distributed periodically varying heat sources have been studied by Banik et al. [47], and for functionally graded material without energy dissipation, has been studied by Mallik and Kanoria [48]

The present work deals with an one-dimensional disturbance in an infinite isotropic functionally graded medium in the context of magneto-thermoelasticity with energy dissipation (GN model type II) and without energy dissipation (GN model type III) in the presence of distributed periodically varying heat sources. The material properties of the FGM are assumed to vary exponentially with the space variable. The governing equations are expressed in a Laplace–Fourier transform domain. The solution for displacement, temperature, stress, and strain in the Laplace transform domain are obtained by taking the Fourier inversion which is carried out by using residual calculus, where the poles of the integrand are obtained numerically in the complex domain by using Leguerre’s method. The numerical inversion of the Laplace transform is done by using a method based on a Fourier series expansion technique (Honig and Hirdes [49]). The results obtained theoretically have been compared numerically and are presented graphically for a copper-like material. A complete and comprehensive analysis and comparison of results of different theories are presented, and the effects of non-homogeneity, magnetic field, and damping coefficient on the displacement, temperature, stress, and strain have been shown graphically.

2 Basic Equations

The constitutive equations are

$$\tau_{ij} = 2\mu e_{ij} + [\lambda\Delta - \gamma(T - T_0)]\delta_{ij}, \tag{2.1}$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \Delta = e_{ii}. \tag{2.2}$$

Stress equations of motion in the presence of body forces F_i are

$$\tau_{ij,j} + F_i = \rho \ddot{u}_i. \tag{2.3}$$

The heat equation corresponding to generalized thermoelasticity with energy dissipation is

$$\rho c_v \ddot{T} + \gamma T_0 \ddot{\Delta} = K \bigtriangledown^2 \dot{T} + K^* \bigtriangledown^2 T + \rho \dot{Q}, \tag{2.4}$$

where $\gamma = (3\lambda + 2\mu)\alpha_{\mathrm{t}}$, K is the thermal conductivity, and K^* is a material constant.

With the effects of a functionally graded solid, the parameters λ, μ, K^*, K, γ, and ρ are no longer constants but are space dependent. Thus, we replace λ, μ, K^*, K, γ, and ρ by $\lambda_0 f(\mathbf{x})$, $\mu_0 f(\mathbf{x})$, $K_0^* f(\mathbf{x})$, $K_0 f(\mathbf{x})$, $\gamma_0 f(\mathbf{x})$ and $\rho_0 f(\mathbf{x})$ where λ_0, μ_0, K_0^*, K_0, γ_0, and ρ_0 are assumed to be constants and $f(\mathbf{x})$ is a given dimensionless function of the space variable $\mathbf{x} = (x, y, z)$. Then the equations corresponding to Eqs. 2.1, 2.3, and 2.4 take the following form:

$$\tau_{ij} = f(\mathbf{x})[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(T - T_0)\}\delta_{ij}]. \tag{2.5}$$

$$\begin{aligned} f(\mathbf{x})\rho_0\ddot{u}_i &= f(\mathbf{x})[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(T - T_0)\}\delta_{ij}]_{,j} \\ &\quad + f(\mathbf{x})_{,j}[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(T - T_0)\}\delta_{ij}] + F_i, \end{aligned} \tag{2.6}$$

and

$$[K_0 f(\mathbf{x})\dot{T}_{,i} + K_0^* f(\mathbf{x})T_{,i}]_{,i} + \rho_0 f(\mathbf{x})\dot{Q} = \rho_0 f(\mathbf{x}) c_v \ddot{T} + \gamma_0 f(\mathbf{x}) T_0 \ddot{\Delta}. \tag{2.7}$$

3 Formulation of the Problem

We now consider a functionally graded infinite isotropic thermoelastic body at a uniform reference temperature T_0 in the presence of periodically varying heat sources distributed over a plane area. We shall consider a one-dimensional disturbance of the medium, so that the displacement vector $\mathbf{u}$ and temperature field T can be expressed in the following form:

$$
\begin{aligned}
\mathbf{u} &= (u(x,t), 0, 0),\\
T &= T(x,t).
\end{aligned} \tag{3.1}
$$

The electromagnetic field is governed by Maxwell's equations (in the absence of the displacement current and charge density) as

$$
\text{curl } \mathbf{H} = \mathbf{J},\ \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \text{div } \mathbf{B} = 0,\ \mathbf{B} = \mu_e \mathbf{H}. \tag{3.2}
$$

The generalized Ohm's law in deformable continua is

$$
\mathbf{J} = \sigma(\mathbf{E} + \dot{\mathbf{u}} \times \mathbf{B}), \tag{3.3}
$$

where the small effect of a temperature gradient on the conduction current $\mathbf{J}$ is neglected.

It is assumed that the material properties depend only on the x-coordinate. So, we take $f(\mathbf{x})$ as $f(x)$. In the context of the linear theory of generalized thermoelasticity based on the Green–Naghdi model III, the equation of motion, heat equation, and constitutive equation can be written as

$$
\begin{aligned}
&f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial^2 u}{\partial x^2} - \gamma_0 \frac{\partial T}{\partial x}\right] + \left[(\lambda_0 + 2\mu_0)\frac{\partial u}{\partial x} - \gamma_0 (T - T_0)\right]\frac{\partial f(x)}{\partial x} + F_x\\
&= \rho_0 f(x) \frac{\partial^2 u}{\partial t^2},
\end{aligned} \tag{3.4}
$$

where

$$
\begin{aligned}
&\mathbf{F} = (\mathbf{J} \times \mathbf{B}), \mathbf{F} = (F_x, F_y, F_z)\\
&\quad \frac{\partial}{\partial x}\left[K_0^* f(x)\frac{\partial T}{\partial x} + K_0 f(x)\frac{\partial \dot{T}}{\partial x}\right] + \rho_0 f(x)\dot{Q}\\
&= \rho_0 f(x) c_v \ddot{T} + \gamma_0 f(x) T_0 \frac{\partial^3 u}{\partial t^2 \partial x},
\end{aligned} \tag{3.5}
$$

$$
\tau_{xx} = f(x)[(\lambda_0 + 2\mu_0)e_{xx} - \gamma_0 (T - T_0)], \tag{3.6}
$$

where

$$
e_{xx} = \frac{\partial u}{\partial x}. \tag{3.7}
$$

We set $\mathbf{H} = \mathbf{H_0} + \mathbf{h}$, where $\mathbf{H_0} = (0, 0, H_0)$. The perturbed magnetic field $\mathbf{h}$ is so small that the product of $\mathbf{h}$ and $\mathbf{u}$ and their derivatives can be neglected for linearization of the field equations.

We assume that all the vector and scalar functions depend only on the spatial coordinate x and time t and are independent of the y and z coordinates.

Equation 3.2$_1$ gives

$$J_x = 0,\ J_y = -\frac{\partial H_z}{\partial x},\ J_z = \frac{\partial H_y}{\partial x}, \tag{3.8}$$

where $\mathbf{J} = (J_x, J_y, J_z), \mathbf{H} = (H_x, H_y, H_z)$.

Equation 3.2$_2$ yields

$$\frac{\partial H_x}{\partial t} = 0,\ \frac{\partial E_z}{\partial x} = \mu_e \frac{\partial H_y}{\partial t},\ \frac{\partial E_y}{\partial x} = -\mu_e \frac{\partial H_z}{\partial t}, \mathbf{E} = (E_x, E_y, E_z). \tag{3.9}$$

Equation 3.2$_3$ gives $\frac{\partial h_x}{\partial x} = 0$ which implies that $h_x = 0$, since initially no perturbed field is applied along the x-axis.

The modified Ohm's law gives

$$J_x = \sigma E_x,\ J_y = \sigma \left[E_y - \mu_e H_z \frac{\partial u}{\partial t}\right],\ J_z = \sigma \left[E_z + \mu_e H_y \frac{\partial u}{\partial t}\right]. \tag{3.10}$$

Now $J_x = 0$ implies $E_x = 0$.

By eliminating J_x, J_y, J_z and using Eqs. 3.2, 3.3, and 3.10, we get

$$\frac{\partial H_z}{\partial t} = \nu_H \frac{\partial^2 H_z}{\partial x^2} - \frac{\partial}{\partial x}(H_z \frac{\partial u}{\partial t}), \tag{3.11}$$

$$\frac{\partial H_y}{\partial t} = \nu_H \frac{\partial^2 H_y}{\partial x^2} - \frac{\partial}{\partial x}(H_y \frac{\partial u}{\partial t}), \tag{3.12}$$

where $\nu_H = (\sigma \mu_e)^{-1}$ is called the magnetic viscosity.

Equation 3.4 reduces to

$$f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial^2 u}{\partial x^2} - \gamma_0 \frac{\partial T}{\partial x}\right] + \left[(\lambda_0 + 2\mu_0)\frac{\partial u}{\partial x} - \gamma_0 (T - T_0)\right]\frac{\partial f(x)}{\partial x}$$
$$- \frac{\partial}{\partial x}\left[\frac{1}{2}\mu_e (H_y^2 + H_z^2)\right] = \rho_0 f(x) \frac{\partial^2 u}{\partial t^2} \tag{3.13}$$

and Eq. 3.5 can be written as

$$\frac{\partial}{\partial x}\left[K_0^* f(x) \frac{\partial T}{\partial x} + K_0 f(x) \frac{\partial^2 T}{\partial x \partial t}\right] + \rho_0 f(x) \dot{Q}$$
$$= \rho_0 f(x) c_v \frac{\partial^2 T}{\partial t^2} + \gamma_0 f(x) T_0 \frac{\partial^3 u}{\partial x \partial t^2}. \tag{3.14}$$

We set $H_z = H_0 + h_z$ where the perturbed magnetic field h_z is small compared to the strong initial magnetic field H_0.

Then from Eqs. 3.11–3.14 after linearization, we get

$$\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} - H_0 \frac{\partial^2 u}{\partial x \partial t}, \ \frac{\partial h_y}{\partial t} = \nu_H \frac{\partial^2 h_y}{\partial x^2} \tag{3.15}$$

and

$$f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial^2 u}{\partial x^2} - \gamma_0 \frac{\partial T}{\partial x}\right] + \left[(\lambda_0 + 2\mu_0)\frac{\partial u}{\partial x} - \gamma_0 (T - T_0)\right]\frac{\partial f(x)}{\partial x}$$
$$- \mu_e H_0 \frac{\partial h_z}{\partial x} = \rho_0 f(x) \frac{\partial^2 u}{\partial t^2}. \tag{3.16}$$

Now for a perfect electrical conductor, $\nu_H \longrightarrow 0$ as $\sigma \longrightarrow \infty$. Equation 3.15_1 leads to $h_z = -H_0 \frac{\partial u}{\partial x}$, since there is no perturbation at ∞. Then Eq. 3.16 reduces to

$$f(x)\left[c_1^2 (1 + R_H)\frac{\partial^2 u}{\partial x^2} - \frac{\gamma_0}{\rho_0}\frac{\partial T}{\partial x}\right] + \left[c_1^2 \frac{\partial u}{\partial x} - \frac{\gamma_0}{\rho_0}(T - T_0)\right]\frac{\partial f(x)}{\partial x} = f(x)\frac{\partial^2 u}{\partial t^2}, \tag{3.17}$$

where $R_H = \frac{\mu_e H_0^2}{\rho_0 c_1^2} = \frac{v_A^2}{c_1^2}$, $c_1 = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}$, and $v_A = \sqrt{\frac{\mu_e}{\rho_0}} H_0$ is the Alf'ven wave velocity of the medium. The coefficient R_H represents the effect of an external magnetic field in the thermoelastic processes proceeding in the body.

We introduce the following dimensionless quantities:

$$x' = \frac{x}{l}, \quad u' = \frac{\lambda_0 + 2\mu_0}{\gamma_0 T_0 l} u, \quad t' = \frac{c_1 t}{l}, \quad \theta = \frac{T - T_0}{T_0},$$
$$f(x') = f(x), \quad \tau'_{x'x'} = \frac{\tau_{xx}}{\gamma_0 T_0}, \quad e'_{x'x'} = e_{xx}, \quad 1 + R_H = R_M^2,$$

where l = some standard length and $c_1 = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}$ is the standard speed, and omitting primes, Eqs. 3.17, 3.14, 3.6, and 3.7 can be re-written in dimensionless form as

$$f(x)\left[R_M^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x}\right] + \left[\frac{\partial u}{\partial x} - \theta\right]\frac{\partial f(x)}{\partial x} = f(x)\frac{\partial^2 u}{\partial t^2}, \tag{3.18}$$

$$c_T^2 \frac{\partial}{\partial x}\left[f(x)\frac{\partial \theta}{\partial x}\right] + \kappa_0 \frac{\partial}{\partial x}\left[f(x)\frac{\partial^2 \theta}{\partial x \partial t}\right] + f(x) Q_0 = f(x)\frac{\partial^2 \theta}{\partial t^2} + \epsilon_T f(x)\frac{\partial^3 u}{\partial x \partial t^2}, \tag{3.19}$$

$$\tau_{xx} = f(x)\left[\frac{\partial u}{\partial x} - \theta\right], \tag{3.20}$$

$$e_{xx} = \frac{\gamma_0 T_0}{\lambda_0 + 2\mu_0}\frac{\partial u}{\partial x}, \tag{3.21}$$

where

$$c_T^2 = \frac{K_0^*}{\rho_0 c_v c_1^2}, \quad \epsilon_T = \frac{\gamma_0^2 T_0}{(\lambda_0 + 2\mu_0)\rho_0 c_v}, \quad \kappa_0 = \frac{K_0}{\rho_0 c_v c_1 l}, \quad Q_0 = \frac{l}{T_0 c_v c_1}\frac{\partial Q}{\partial t}.$$

We assume that the medium is initially at rest. The undisturbed state is maintained at reference a temperature. Then we have

$$u(x, 0) = \dot{u}(x, 0) = \theta(x, 0) = \dot{\theta}(x, 0) = 0. \tag{3.22}$$

3.1 Exponential Variation of Non-Homogeneity

We take $f(x) = \mathrm{e}^{-nx}$, where n is a dimensionless constant.Then the corresponding equations, Eqs. 3.18–3.21, reduce to

$$R_M^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} - n\left(\frac{\partial u}{\partial x} - \theta\right) = \frac{\partial^2 u}{\partial t^2}, \tag{3.23}$$

$$c_T^2\left(\frac{\partial^2 \theta}{\partial x^2} - n\frac{\partial \theta}{\partial x}\right) + \kappa_0\left(\frac{\partial^3 \theta}{\partial x^2 \partial t} - n\frac{\partial^2 \theta}{\partial x \partial t}\right) + Q_0 = \epsilon_T \frac{\partial^3 u}{\partial t^2 \partial x} + \frac{\partial^2 \theta}{\partial t^2}, \tag{3.24}$$

$$\tau_{xx}(x, t) = \mathrm{e}^{-nx}\left(\frac{\partial u}{\partial x} - \theta\right), \tag{3.25}$$

$$e_{xx}(x, t) = \beta_1 \frac{\partial u}{\partial x}. \tag{3.26}$$

where

$$\beta_1 = \frac{\gamma_0 T_0}{\lambda_0 + 2\mu_0}.$$

Let us define the Laplace–Fourier double transform of the function $g(x, t)$ by

$$\bar{g}(x, p) = \int_0^\infty g(x, t)\mathrm{e}^{-pt}\,\mathrm{d}t, \quad \mathrm{Re}(p) > 0$$

$$\hat{\bar{g}}(\alpha, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{g}(x, p)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x.$$

Applying the Laplace–Fourier double integral transform to Eqs. 3.23–3.26 and using the relation in Eq. 3.20 we get

$$(R_M^2\alpha^2 + p^2 - \mathrm{i}n\alpha)\hat{\bar{u}}(\alpha, p) = (\mathrm{i}\alpha + n)\hat{\bar{\theta}}(\alpha, p), \tag{3.27}$$

$$[c_T^2\alpha(\alpha - \mathrm{i}n) + p^2 + \alpha p\kappa_0(\alpha - \mathrm{i}n)]\hat{\bar{\theta}}(\alpha, p) = \mathrm{i}\epsilon_T \alpha p^2 \hat{\bar{u}}(\alpha, p) + \hat{\bar{Q}}_0, \tag{3.28}$$

$$\hat{\bar{\tau}}_{xx}(\alpha, p) = -\mathrm{i}(\alpha + \mathrm{i}n)\hat{\bar{u}}(\alpha + \mathrm{i}n, p) - \hat{\bar{\theta}}(\alpha + \mathrm{i}n, p), \tag{3.29}$$

$$\hat{\bar{e}}_{xx} = -\mathrm{i}\alpha\beta_1\hat{\bar{u}}(\alpha, p). \tag{3.30}$$

Solving Eqs. 3.27 and 3.28 for $\hat{\bar{u}}(\alpha, p)$ and $\hat{\bar{\theta}}(\alpha, p)$, we get

$$\hat{\bar{u}}(\alpha, p) = \frac{\hat{\bar{Q}}_0(\mathrm{i}\alpha + n)}{M(\alpha)}, \tag{3.31}$$

$$\hat{\bar{\theta}}(\alpha, p) = \frac{\hat{\bar{Q}}_0(R_M^2\alpha^2 + p^2 - \mathrm{i}\alpha n)}{M(\alpha)}, \tag{3.32}$$

where

$$\begin{aligned} M(\alpha) &= \left(c_T^2 + p\kappa_0\right) R_M^2\alpha^4 - \alpha^3\left(2\mathrm{i}c_T^2 n + \mathrm{i}np\kappa_0 + \mathrm{i}np\kappa_0 R_M^2\right) \\ &\quad + \alpha^2\left[p^2(R_M^2 + \epsilon_T + c_T^2) + p^3\kappa_0 - c_T^2 n^2 - n^2 p\kappa_0\right] \\ &\quad - \alpha\left[p^2\mathrm{i}n(1 + \epsilon_T + c_T^2 + p\kappa_0)\right] + p^4 \\ &= \left(c_T^2 + p\kappa_0\right) R_M^2(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha - \alpha_4). \end{aligned} \tag{3.33}$$

Now the expressions for the stress and strain in the Laplace–Fourier transform domain can be obtained from Eqs. 3.29 and 3.30 using Eqs. 3.31 and 3.32

$$\begin{aligned} \hat{\bar{\tau}}_{xx}(\alpha, p) &= \frac{\hat{\bar{Q}}_0(1 - R_M^2)(\alpha + \mathrm{i}n)^2}{M(\alpha + \mathrm{i}n)} - \frac{p^2\hat{\bar{Q}}_0}{M(\alpha + \mathrm{i}n)} \\ &= \frac{\hat{\bar{Q}}_0\left[(1 - R_M^2)(\alpha + \mathrm{i}n)^2 - p^2\right]}{M(\alpha + \mathrm{i}n)}, \end{aligned} \tag{3.34}$$

$$\hat{\bar{e}}_{xx}(\alpha, p) = \frac{\beta_1\hat{\bar{Q}}_0\alpha(\alpha - \mathrm{i}n)}{M(\alpha)}. \tag{3.35}$$

Thus, the solution for the displacement, temperature, stress, and strain in the Laplace transform domain can be obtained in terms of the following four integrals:

$$\bar{u}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{Q}}_0(\mathrm{i}\alpha + n)}{M(\alpha)}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \tag{3.36}$$

$$\bar{\theta}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{Q}}_0\left(R_M^2\alpha^2 + p^2 - \mathrm{i}n\alpha\right)}{M(\alpha)}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \tag{3.37}$$

$$\bar{\tau}_{xx}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{Q}}_0\left[(1 - R_M^2)(\alpha + \mathrm{i}n)^2 - p^2\right]}{M(\alpha + \mathrm{i}n)}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \tag{3.38}$$

$$\bar{e}_{xx}(x, p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\beta_1 \hat{\bar{Q}}_0 \alpha(\alpha - \mathrm{i}n)}{M(\alpha)} \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha. \tag{3.39}$$

where,

$$\begin{aligned} M(\alpha + \mathrm{i}n) &= \left(c_T^2 + p\kappa_0\right) R_M^2 \alpha^4 + \mathrm{i}n \left(4c_T^2 R_M^2 + 3\kappa_0 p R_M^2 - 2c_T^2 - p\kappa_0\right) \alpha^3 \\ &\quad - (6c_T^2 n^2 R_M^2 + 3\kappa_0 n^2 p R_M^2 - 5c_T^2 n^2 - 2\kappa_0 n^2 p - c_T^2 p^2 - \epsilon_T p^2 \\ &\quad - \kappa_0 p^3 - p^2 R_M^2)\alpha^2 - \mathrm{i}n(4c_T^2 n^2 R_M^2 + \kappa_0 n^2 p R_M^2 - 4c_T^2 n^2 \\ &\quad - n^2 \kappa_0 p + p^2 - c_T^2 p^2 - \epsilon_T p^2 - \kappa_0 p^3 - 2p^2 R_M^2)\alpha - c_T^2 n^4 \\ &\quad + n^2 p^2 + c_T^2 n^4 R_M^2 - n^2 p^2 R_M^2 + p^4 \\ &= (c_T^2 + p\kappa_0) R_M^2 (l - l_1)(l - l_2)(l - l_3)(l - l_4). \end{aligned} \tag{3.40}$$

3.2 Periodically Varying Heat Source

Now let us assume that the heat source is distributed over the plane $x = 0$ in the following form:

$$\begin{aligned} Q_0 &= Q_0^* \delta(x) \sin \frac{\pi t}{\tau}, \quad 0 \le t \le \tau \\ &= 0. \qquad\qquad\qquad t > \tau \end{aligned} \tag{3.41}$$

Then,

$$\hat{\bar{Q}}_0 = \frac{Q_0^* \pi \tau (1 + \mathrm{e}^{-p\tau})}{\sqrt{2\pi}(\pi^2 + p^2 \tau^2)}. \tag{3.42}$$

Thus, the expressions for the displacement, temperature, stress, and strain in the Laplace transform domain take the following form:

$$\bar{u}(x, p) = \int_{-\infty}^{\infty} \frac{Q_0^* \tau (1 + \mathrm{e}^{-p\tau})(\mathrm{i}\alpha + n)}{2(\pi^2 + p^2\tau^2) M(\alpha)} \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha, \tag{3.43}$$

$$\bar{\theta}(x, p) = \int_{-\infty}^{\infty} \frac{Q_0^* \tau (1 + \mathrm{e}^{-p\tau}) \left(R_M^2 \alpha^2 + P^2 - \mathrm{i}n\alpha\right)}{2(\pi^2 + p^2\tau^2) M(\alpha)} \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha, \tag{3.44}$$

$$\bar{\tau}_{xx}(x, p) = \int_{-\infty}^{\infty} \frac{Q_0^* \tau (1 + \mathrm{e}^{-p\tau}) \left[\left(1 - R_M^2\right)(\alpha + \mathrm{i}n)^2 - p^2\right]}{2(\pi^2 + p^2\tau^2) M(\alpha + \mathrm{i}n)} \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha, \tag{3.45}$$

$$\bar{e}_{xx}(x, p) = \int_{-\infty}^{\infty} \frac{\beta_1 Q_0^* \tau (1 + e^{-p\tau}) \alpha (\alpha - in)}{2(\pi^2 + p^2\tau^2) M(\alpha)} e^{-i\alpha x}\, d\alpha. \tag{3.46}$$

Applying contour integration to the Eqs. 3.43–3.46 we obtain

$$\begin{aligned}
\bar{u}(x, p) &= -\frac{iQ_0^* \pi \tau (1 + e^{-p\tau})}{R_M^2 (c_T^2 + p\kappa_0)(\pi^2 + p^2\tau^2)} \\
&\quad \times \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j (i\alpha_j + n) e^{-i\alpha_j x} \qquad \text{for } x > 0 \\
&= \frac{iQ_0^* \pi \tau (1 + e^{-p\tau})}{R_M^2 (c_T^2 + p\kappa_0)(\pi^2 + p^2\tau^2)} \\
&\quad \times \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j (i\alpha_j + n) e^{-i\alpha_j x}, \qquad \text{for } x < 0
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
\bar{\theta}(x, p) &= -\frac{iQ_0^* \pi \tau (1 + e^{-p\tau})}{R_M^2 (c_T^2 + p\kappa_0)(\pi^2 + p^2\tau^2)} \\
&\quad \times \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j \left(R_M^2 \alpha_j^2 + p^2 - in\alpha_j \right) e^{-i\alpha_j x} \qquad \text{for } x > 0 \\
&= -\frac{iQ_0^* \pi \tau (1 + e^{-p\tau})}{R_M^2 (c_T^2 + p\kappa_0)(\pi^2 + p^2\tau^2)} \\
&\quad \times \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j \left(R_M^2 \alpha_j^2 + p^2 - in\alpha_j \right) e^{-i\alpha_j x}, \qquad \text{for } x < 0
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\bar{\tau}_{xx}(x, p) &= -\frac{iQ_0^* \pi \tau (1 + e^{-p\tau})}{R_M^2 (c_T^2 + p\kappa_0)(\pi^2 + p^2\tau^2)} \\
&\quad \times \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)<0}}^{4} B_j \left[\left(1 - R_M^2\right) \left(l_j + in\right)^2 - p^2 \right] e^{-i\alpha_j x} \text{ for } x > 0 \\
&= \frac{iQ_0^* \pi \tau (1 + e^{-p\tau})}{R_M^2 (c_T^2 + p\kappa_0)(\pi^2 + p^2\tau^2)} \\
&\quad \times \sum_{j=1\,\mathrm{Im}(\alpha_j)>0}^{4} B_j \left[\left(1 - R_M^2\right) \left(l_j + in\right)^2 - p^2 \right] e^{-i\alpha_j x}, \quad \text{for } x < 0
\end{aligned} \tag{3.49}$$

$$\begin{aligned}\bar{e}_{xx}(x,p) &= -\frac{\mathrm{i}\beta_1 Q_0^*\pi\tau(1+\mathrm{e}^{-p\tau})}{R_M^2(c_T^2+p\kappa_0)(\pi^2+p^2\tau^2)}\\ &\quad\times\sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j\alpha_j(\alpha_j-\mathrm{i}n)\mathrm{e}^{-\mathrm{i}\alpha_j x}\quad \text{for}\ \ x>0\\ &= \frac{\mathrm{i}\beta_1 Q_0^*\pi\tau(1+\mathrm{e}^{-p\tau})}{R_M^2(c_T^2+p\kappa_0)(\pi^2+p^2\tau^2)}\\ &\quad\times\sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j\alpha_j(\alpha_j-\mathrm{i}n)\mathrm{e}^{-\mathrm{i}\alpha_j x},\quad \text{for}\ \ x<0\end{aligned} \tag{3.50}$$

where A_j's and B_j's are given by

$$\begin{aligned}A_j &= \prod_{\substack{n=1\\ n\neq j}}^{4}\frac{1}{(\alpha_j-\alpha_n)}\\ B_j &= \prod_{\substack{n=1\\ n\neq j}}^{4}\frac{1}{(l_j-l_n)}\quad j=1,2,3,4\end{aligned} \tag{3.51}$$

4 Inversion of Laplace Transform

It is difficult to find the inverse Laplace transform of the complicated solutions for the displacement, temperature, stress, and strain in the Laplace transform domain. So we have to resort to numerical computation. We now outline the numerical procedure to solve the problem. Let $\bar{f}(x,p)$ be the Laplace transform of a function $f(x,t)$.

Then the inversion formula for Laplace transform can be written as

$$f(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \mathrm{e}^{pt}\bar{f}(x,p)\,\mathrm{d}p \tag{4.1}$$

where d is an arbitrary real number greater than real parts of all singularities of $\bar{f}(x,p)$.

Taking $p=d+\mathrm{i}w$, the preceding integral takes the form,

$$f(x,t) = \frac{\mathrm{e}^{dt}}{2\pi}\int_{-\infty}^{\infty} f(x,d+\mathrm{i}w)\,\mathrm{d}w. \tag{4.2}$$

Expanding the function $h(x,t) = \mathrm{e}^{-dt} f(x,t)$ in a Fourier series in the interval $[0, 2T]$, we obtain the approximate formula (Honig and Hirdes [49])

$$f(x,t) = f_\infty(x,t) + E_\mathrm{D} \tag{4.3}$$

where

$$f_\infty(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_\mathrm{K} \quad 0 \le t \le 2T \tag{4.4}$$

$$c_\mathrm{k} = \frac{\mathrm{e}^{\mathrm{d}t}}{T}\left[\mathrm{e}^{\frac{\mathrm{i}k\pi t}{T}} \bar{f}(x, d + \frac{\mathrm{i}k\pi t}{T})\right]. \tag{4.5}$$

The discretization error E_D can be made arbitrarily small by choosing d large enough (Honig and Hirdes [49]). Since the infinite series in Eq. 4.4 can be summed upto a finite number N of terms, the approximate value $f(x,t)$ becomes

$$f_N(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_\mathrm{k}, \quad 0 \le t \le 2T \tag{4.6}$$

Using the preceding formula to evaluate $f(x,t)$, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First the "Korrecktur" method is applied to reduce the discretization error. Next the ϵ-algorithm is used to accelerate convergence (Honig and Hired [49]).

The Korrecktur method uses the following formula to evaluate the function $f(x,t)$:

$$f(x,t) = f_\infty(x,t) - \mathrm{e}^{-2dT} f_\infty(x, 2T + t) + E'_\mathrm{D}, \tag{4.7}$$

where the discretization error $\mid E'_\mathrm{D} \mid \le \mid E_\mathrm{D} \mid$. Thus, the approximate value of $f(x,t)$ becomes

$$f_{NK}(x,t) = f_N(x,t) - \mathrm{e}^{-2dT} f_{N'}(x, 2T + t), \tag{4.8}$$

where N' is an integer such that $N' < N$.

We shall now describe the ϵ-algorithm that is used to accelerate the convergence of the series in Eq. 4.6. Let $N = 2q + 1$, where q is a natural number and $s_m = \sum_{k=1}^{m} c_\mathrm{k}$ is the sequence of the partial sum of the series in Eq. 4.6

We define ϵ-sequence by

$$\epsilon_{0,m} = 0, \epsilon_{1,m} = s_m$$

and

$$\epsilon_{p+1,m} = \epsilon_{p-1,m} + \frac{1}{\epsilon_{p,m+1} - \epsilon_{p,m}}, \quad p = 1, 2, 3, \ldots$$

It can be shown that (Honig and Hirdes 1984 [49]) the sequence $\epsilon_{1,1}, \epsilon_{3,1}, \epsilon_{5,1}, \ldots,$ $\epsilon_{N,1}$ converges to $f(x,t) + E_{\mathrm{D}} - \dfrac{c_0}{2}$ faster than the sequence of partial sums s_m, $m = 1, 2, 3, \ldots$

The actual procedure used to invert the Laplace transform consists of using Eq. 4.8 together with the ϵ-algorithm. The values of d and T are chosen according to the criteria outlined in [49].

5 Numerical Result and Discussion

To get the solution for the thermal displacement, temperature, stress, and strain in the space-time domain, we have to apply the Laplace inversion formula to Eqs. 3.47–3.50, respectively. This has been done numerically using a method based on the Fourier series expansion technique mentioned above. To get the roots of the polynomial $M(\alpha)$ and $M(\alpha + \mathrm{i}n)$ in the complex domain, we have used Laguerre's method. The numerical code has been prepared using Fortran77 programming language. For computational purposes, a copper-like material with a material constant (Roychoudhui and Dutta [25]) has been taken into consideration.

$$\epsilon_T = 0.0168,\ \lambda = 1.387 \times 10^{11}\,\mathrm{N{\cdot}m^{-2}},\ \mu = 0.448 \times 10^{11}\,\mathrm{N{\cdot}m^{-2}},$$
$$\alpha_t = 1.67 \times 10^{-8}\,{}^{\circ}\mathrm{C}^{-1}, \theta = 1\,{}^{\circ}\mathrm{C}$$

Also, we have taken $Q_0^* = 1,\ \tau = 1,\ c_P = 1$, and $c_T = 2$ so the faster wave is the thermal wave.

We now present our results in the form of graphs (Figs. 1, 2, 3, 4, 5, 6, 7, 8) to compare the thermal displacement, temperature, thermal stress, and strain in the case of the TEWOED and TEWED models under the influence of a magnetic field for a functionally graded material.

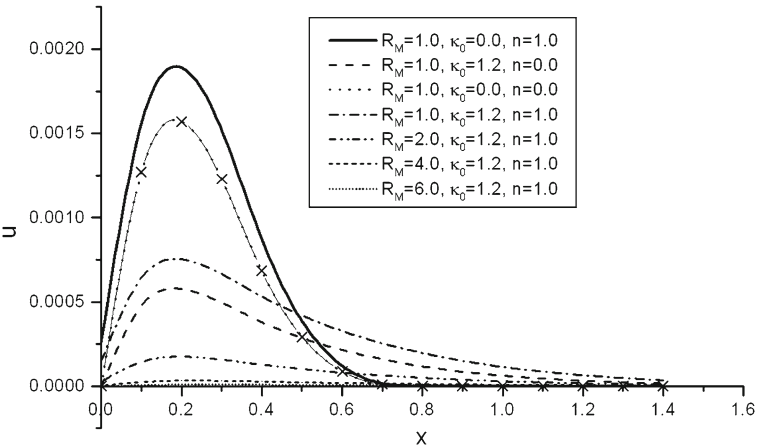

Fig. 1 Variation of displacement with distance x for $t = 0.4$, $\times$ estimated from Roychoudhuri and Dutta [25]

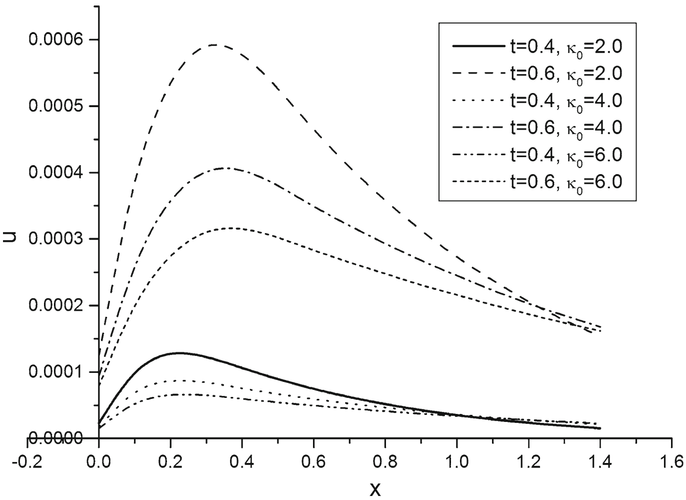

Fig. 2 Variation of displacement with distance x for $n = 1.0$ and $R_M = 2.0$

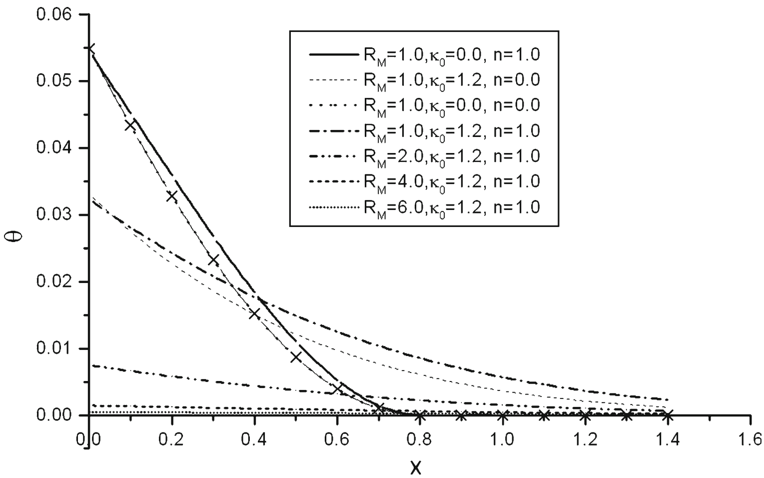

Fig. 3 Variation of temperature θ with distance x for $t = 0.4$, $\times$ estimated from Roychoudhuri and Dutta [25]

Figure 1 depicts the variation of the thermal displacement (u) versus distance x for time $t = 0.4$. It is observed that the displacement increases for $0.0 \leq x \leq 0.2$ in the absence of both the magnetic field and dissipation of energy ($R_M = 1.0, \kappa_0 = 0.0$) and then decreases and ultimately goes to zero for $x \geq 0.8$, where $n = 1.0$. The result absolutely complies with that of Mallik and Kanoria [48]). Also, in the case when ($n = 0.0$), u increases at first, then decreases and ultimately disappears, as before, with the increase of x. As may be seen from the figure, $\kappa_0 = 1.2$ corresponds to a slower rate of decay than the case when $\kappa_0 = 0.0$. The result agrees with that of Banik et al. [47]. The comparison of the results obtained in [48] and [47] and by the

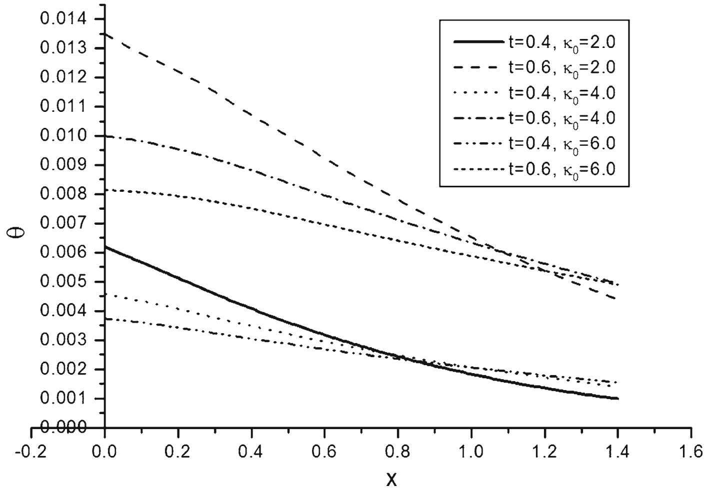

Fig. 4 Variation of temperature θ with distance x for $n = 1.0$ and $R_M = 2.0$

present numerical methods are given in Table 1. Moreover, for ($R_M = 1.0$), ($\kappa_0 = 0.0$), and ($n = 0.0$), the result obtained by the present numerical method is shown in Fig. 1 by dots (...) and the estimated result by Roychoudhuri and Dutta [25] is shown by cross ($\times$) in the same figure where they have used the analytical method. The nature of the profile of u is observed by taking $R_M = 2, 4, 6$ and keeping $\kappa_0 = 1.2$ and $n = 1.0$. It is observed that with the increase of the magnetic field, the magnitude of the displacement decreases which is quite plausible. It is also clear from Table 1.

Figure 2 depicts the variation of the thermal displacement with distance taking the non-homogeneous parameter $n = 1.0$, the magnetic field $R_M = 2.0$, and $t = 0.4, 0.6$, where we have considered the GN III model (TEWED). Now it is observed that, as the damping coefficient increases, the rate of decay of the displacement becomes slow and, for $t = 0.6$, the magnitude of the thermal displacement is greater than the magnitude of the same for $t = 0.4$ for a particular value of x.

Figure 3 is plotted to show the variation of temperature θ with distance x for time $t = 0.4$. Figure 3 depicts the effect the of magnetic field on the temperature when the non-homogeneous parameter $n = 1.0$ and there is a dissipation energy ($\kappa_0 = 1.2$). Here also a similar qualitative behavior is observed as in the case of Fig. 1. This can also be verified from the expression of $\bar{\theta}$ given in Eq. 3.48 involving $e^{-i\alpha_j x}$, $\mathrm{Im}(\alpha_j) < 0$ for $x \geq 0$.

Figure 4 depicts the variation of temperature with distance for the non-homogeneous parameter $n = 1.0$ and $R_M = 2.0$. Here we have considered the GN III model (TEWED), i.e., dissipation of energy has occurred. It is observed that as the damping coefficient increases, i.e., $\kappa_0 = 2.0, 4.0, 6.0$ and $t = 0.4$, the temperature decreases very slowly with distance and the rate of damping of the temperature increases with the increase of the damping coefficient. But for $t = 0.6$ when $\kappa_0 = 2.0$, the temperature decreases with distance and almost linearly, but for $\kappa_0 = 4.0$ and $\kappa_0 = 6.0$, it decreases more slowly than for the case where $\kappa_0 = 2.0$.

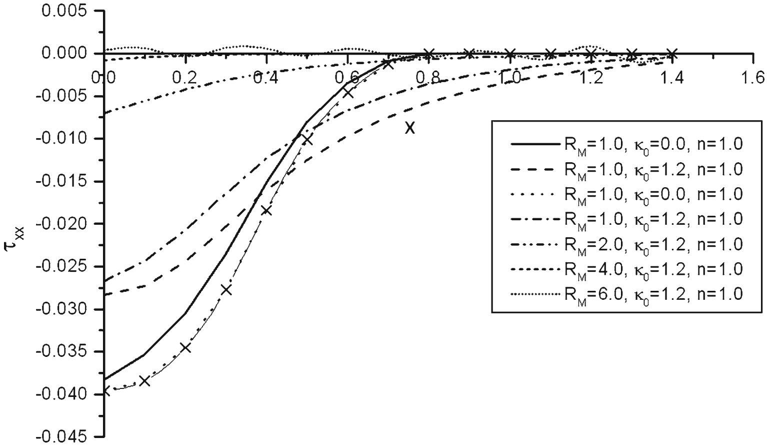

Fig. 5 Variation of stress τ_{xx} with distance x for $t = 0.4$, $\times$ estimated from Roychoudhuri and Dutta [25]

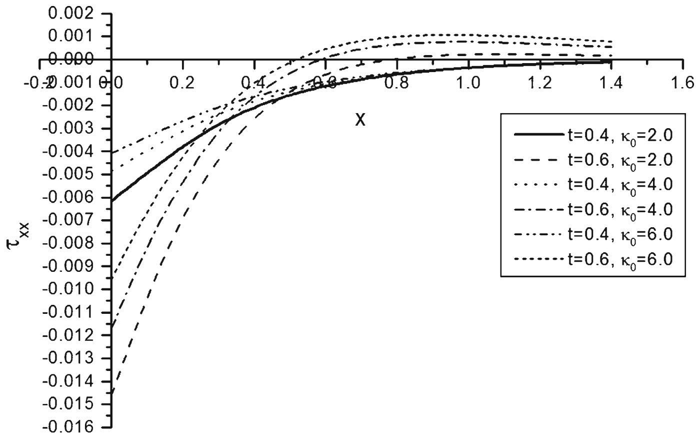

Fig. 6 Variation of stress τ_{xx} with displacement x for $n = 1.0$ and $R_M = 2.0$

Figure 5 shows the variation of thermal stress versus distance x for time $t = 0.4$. This figure depicts the effect of the magnetic field when the non-homogeneous parameter $n = 1.0$ and there is a dissipation of energy (we take the damping coefficient $\kappa_0 = 1.2$). And it is observed that in this case when the magnetic field $R_M = 2.0$, the stress is negative and its magnitude decreases very slowly and finally becomes zero. But when the magnetic field $R_M = 4.0$, the magnitude of the thermal stress tends to zero and for $R_M = 6.0$, the oscillatory nature is observed for small values of x.

Figure 6 depicts the variation of the thermal stress with distance x for the non-homogeneous parameter $n = 1.0$ and $R_M = 2.0$. This is the case using the GN III model (TEWED), i.e., dissipation of energy has occurred. Now it is observed that, for $t = 0.4$, the stress is negative, and as the damping coefficient increases, the magnitude of the stress decreases for $0 < x < 0.6$. For $t = 0.6$, the magnitude of the stress is negative for $0 < x < 0.8$ ($\kappa_0 = 2$), $0 < x < 0.6$ ($\kappa_0 = 4$), and $0 < x < 0.5$ ($\kappa_0 = 6$)

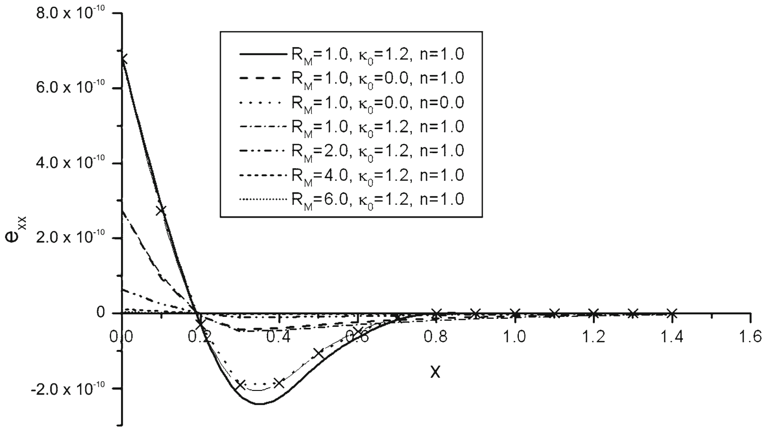

Fig. 7 Variation of strain e_{xx} with distance x for $t = 0.4$, $\times$ estimated from Roychoudhuri and Dutta [25]

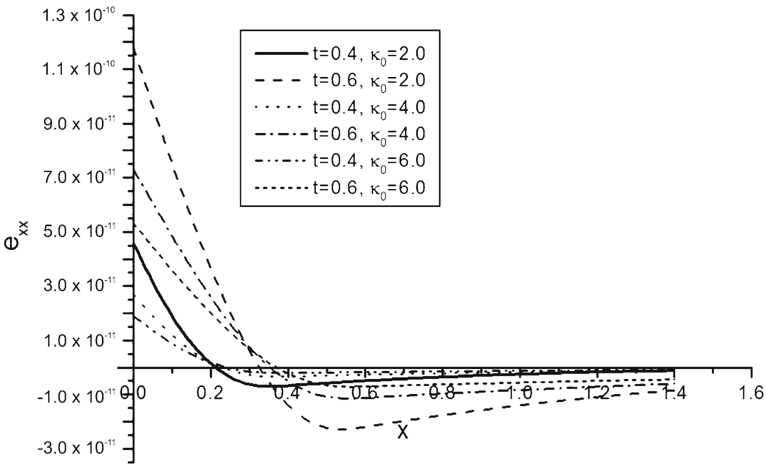

Fig. 8 Variation of strain with distance x for $n = 1.0$ and $R_M = 2.0$

and after this, it is positive and finally diminishes to zero. This is also in conformity with the fact that the stress should decrease with the increasing distance x from the plane $x = 0$, where the heat source is applied.

Figure 7 gives the variation of the thermal strain against distance x for time $t = 0.4$. From this figure we can also show that, for $\kappa_0 = 1.2$ and $n = 1.0$, the strain is positive up to a distance $x = 0.2$ for $R_M = 1, 2, 4$, but as the magnetic field increases, the magnitude of the strain decreases ($R_M = 1, 2, 4$). Then it becomes negative and finally diminishes to zero. The magnitude of the strain almost coincides with the x-axis. This concludes when with an increase of the magnetic field, the deformation in the body is almost nil. This is also clear from Table 2.

Figure 8 depicts the variation of the thermal strain with distance for the non-homogeneous parameter $n = 1.0$, $R_M = 2.0$, $\kappa_0 = 2, 4, 6$, and $t = 0.4, 0.6$. This is the case for the GN III model (TEWED), i.e., dissipation of energy has occurred.

Table 1 Variation of thermal displacement u versus distance x for $t = 0.4$

x	$R_M = 1.0, \kappa_0 = 0.0, n = 1.0$		$R_M = 1.0, \kappa_0 = 1.2, n = 0.0$	
	Results obtained by Mallik and Kanoria [48]	Results obtained by present numerical method	Results obtained by Banik et al. [47]	Results obtained by present numerical method
0.0	0.000263	0.000263	0.000000	0.000000
0.1	0.001560	0.001560	0.000478	0.000478
0.2	0.001890	0.001890	0.000576	0.000576
0.3	0.001520	0.001520	0.000496	0.000496
0.4	0.000873	0.000873	0.000381	0.000381
0.5	0.000388	0.000388	0.000287	0.000287
0.6	0.000122	0.000122	0.000215	0.000215
0.7	0.000016	0.000016	0.000160	0.000160
0.8	0.000000	0.000000	0.000118	0.000118
0.9	0.000000	0.000000	0.000087	0.000087
1.0	0.000000	0.000000	0.000063	0.000063
1.1	0.000000	0.000000	0.000046	0.000046
1.2	0.000000	0.000000	0.000033	0.000033
1.3	0.000000	0.000000	0.000023	0.000023
1.4	0.000000	0.000000	0.000016	0.000016

Table 2 Variation of temperature θ versus distance x for $t = 0.4$

x	$R_M = 1.0, \kappa_0 = 0.0, n = 1.0$		$R_M = 1.0, \kappa_0 = 1.2, n = 0.0$	
	Results obtained by Mallik and Kanoria [48]	Results obtained by present numerical method	Results obtained by Banik et al. [47]	Results obtained by present numerical method
0.0	0.054500	0.054500	0.033100	0.033100
0.1	0.045200	0.045200	0.027500	0.027500
0.2	0.035900	0.035900	0.022700	0.022700
0.3	0.026800	0.026800	0.018600	0.018600
0.4	0.018400	0.018400	0.015100	0.015100
0.5	0.011100	0.011100	0.012100	0.012100
0.6	0.005290	0.005290	0.009700	0.009700
0.7	0.001430	0.001430	0.007690	0.007690
0.8	0.000005	0.000005	0.006050	0.006050
0.9	0.000000	0.000000	0.004720	0.004720
1.0	0.000000	0.000000	0.003650	0.003650
1.1	0.000000	0.000000	0.002800	0.002800
1.2	0.000000	0.000000	0.002130	0.002130
1.3	0.000000	0.000000	0.001610	0.001610
1.4	0.000000	0.000000	0.001210	0.001210

Table 3 Variation of thermal stress τ_{xx} versus distance x for $t = 0.4$

x	$R_M = 1.0, \kappa_0 = 0.0, n = 1.0$		$R_M = 1.0, \kappa_0 = 1.2, n = 0.0$	
	Results obtained by Mallik and Kanoria [48]	Results obtained by present numerical method	Results obtained by Banik et al. [47]	Results obtained by present numerical method
0.0	−0.038300	−0.038300	−0.028300	−0.028300
0.1	−0.035300	−0.035300	−0.027300	−0.027300
0.2	−0.030400	−0.030400	−0.024400	−0.024400
0.3	−0.023500	−0.023500	−0.020300	−0.020300
0.4	−0.015100	−0.015100	−0.016000	−0.016000
0.5	−0.008020	−0.008020	−0.012500	−0.012500
0.6	−0.003480	−0.003480	−0.009660	−0.009660
0.7	−0.000886	−0.000886	−0.007450	−0.007450
0.8	−0.000003	−0.000003	−0.005720	−0.005720
0.9	0.000000	0.000000	−0.004370	−0.004370
1.0	0.000000	0.000000	−0.003320	−0.003320
1.1	0.000000	0.000000	−0.002510	−0.002510
1.2	0.000000	0.000000	−0.001890	−0.001890
1.3	0.000000	0.000000	−0.001410	−0.001410
1.4	0.000000	0.000000	−0.001050	−0.001050

Now it is observed that, for $t = 0.4$, the strain is positive in the range $0 \leq x \leq 0.2$, then becomes negative in the range $0.2 \leq x \leq 0.9$, and finally becomes zero. But, for $t = 0.6$, the stress is positive in the range $0 \leq x \leq 0.31 (\kappa_0 = 2.0)$, in the range $0 \leq x \leq 0.36$ ($\kappa_0 = 4.0$), and in the range $0 \leq x \leq 0.38$ ($\kappa_0 = 6.0$) and then it becomes negative.

In all the figures when there is neither a magnetic field ($R_M = 1.0$) nor a dissipation of energy ($\kappa_0 = 0.0$), the result agrees with that of Mallik and Kanoria [48] for functionally graded materials. For the homogeneous case ($n = 0.0$) and in the absence of a magnetic field ($R_M = 1.0$), the result agrees with that of Banik et al. [47]. The comparison of the results obtained in [48] and [47] and by the present numerical methods is shown in Tables 2, 3, and 4. When $n = 0.0$, $R_M = 1.0$, and $\kappa_0 = 0.0$, the result is confirmed by that of Roychoudhuri and Dutta [25] in which the closed form solution of the problem has been derived and the estimated results from [25] are shown with crosses in the figures (Figs. 3, 5, 7).

6 Conclusions

This paper studies the magneto-thermoelastic interactions in a functionally graded isotropic unbounded medium due to the presence of periodically varying heat sources in the context of the linear theory of generalized thermoelasticity with energy dissipation (TEWED) and without energy dissipation (TEWOED). The material properties

Table 4 Variation of thermal strain e_{xx} versus distance x for $t = 0.4$

x	$R_M = 1.0, \kappa_0 = 0.0, n = 1.0$		$R_M = 1.0, \kappa_0 = 1.2, n = 0.0$	
	Results obtained by Mallik and Kanoria [48]	Results obtained by present numerical method	Results obtained by Banik et al. [47]	Results obtained by present numerical method
0.0	6.8×10^{-10}	6.8×10^{-10}	2.74×10^{-10}	2.74×10^{-10}
0.1	2.89×10^{-10}	2.89×10^{-10}	9.32×10^{-11}	9.32×10^{-11}
0.2	-2.99×10^{-11}	-2.99×10^{-11}	-7.9×10^{-12}	-7.9×10^{-12}
0.3	-2.19×10^{-10}	-2.19×10^{-10}	-4.27×10^{-11}	-4.27×10^{-11}
0.4	-2.25×10^{-10}	-2.25×10^{-10}	-3.91×10^{-11}	-3.91×10^{-11}
0.5	-1.36×10^{-10}	-1.36×10^{-10}	-3.03×10^{-11}	-3.03×10^{-11}
0.6	-6.48×10^{-11}	-6.48×10^{-11}	-2.33×10^{-11}	-2.33×10^{-11}
0.7	-1.76×10^{-11}	-1.76×10^{-11}	-1.77×10^{-11}	-1.77×10^{-11}
0.8	-7.05×10^{-14}	-7.05×10^{-14}	-1.34×10^{-11}	-1.34×10^{-11}
0.9	-9.07×10^{-19}	-9.07×10^{-19}	-1.01×10^{-11}	-1.01×10^{-11}
1.0	1.11×10^{-19}	1.11×10^{-19}	-7.51×10^{-12}	-7.51×10^{-12}
1.1	-1.15×10^{-20}	-1.15×10^{-20}	-5.55×10^{-12}	-5.55×10^{-12}
1.2	3.7×10^{-20}	3.7×10^{-20}	-4.07×10^{-12}	-4.07×10^{-12}
1.3	-9.8×10^{-23}	-9.8×10^{-23}	-2.96×10^{-12}	-2.96×10^{-12}
1.4	-1.67×10^{-21}	-1.67×10^{-21}	-2.14×10^{-12}	-2.14×10^{-12}

under consideration are assumed to vary exponentially with distance. The analysis of the results permits some concluding remarks.

1. The presence of a magnetic field and damping coefficient has a significant effect on the solution of the displacement, temperature, stress, and strain. From the graphs it is clear that with an increase of both the magnetic field and the damping coefficient, the magnitudes of the displacement, temperature, stress, and strain decrease.
2. The results obtained in this paper agree with those of Mallik and Kanoria [48] when both the magnetic field and the dissipation of energy are absent. They also agree with Banik et al. [47], when the homogeneous material in the absence of a magnetic field is considered. Moreover, the solution of Roychoudhuri and Dutta [25] can be derived from the present solution considering the body to be homogeneous in the absence of both the magnetic field and dissipation of energy in which the closed form solution of the problem has been obtained.

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